Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

$$\frac{1}{P} = x^m \log x = \log x^{x^m}$$

The demonstration of Möbius applying to the expression

$$x^{x^m}, \text{ we have when } x = 0 \qquad \frac{1}{p} = \log 1 = 0,$$

but by hypothesis p is a finite quantity. Making use of this function he constructs the example $x \frac{a+x}{\log x}$ which takes the form 0^0 when $x = 0$, but it is equivalent to ε^{a+x}, hence when $x = 0$ its value is ε^a.

The condition under which a function may assume the form 0^0 with a value other than unity may be found as follows: Let u^v be the function, u and v vanishing simultaneously. If u vanish with a finite ratio to any finite power of v, say v^n, let $\left.\frac{u}{v^n}\right]_0 = m$; then, since $v^v]_0 = 1$,

$$u^v]_0 = m^v v^{nv}]_0 = m^0 v^0]_0^n = m^0 . 1^n = 1.$$

But if there is no finite value of n, we fail to establish the equation. Hence the only exceptional cases are those in which

$$\left.\frac{u}{v^n}\right]_0 = 0, \text{ for all finite values of } n.$$

SOLUTIONS OF PROBLEMS IN NUMBER THREE.

Solutions of problems in No. 3 have been received as follows:

From Marcus Baker, 111, 112, 113, 114 and 118; Dr. H. Eggers, 113; [Dr. Eggers also furnished an elegant solution of 109, which was overlooked in our notice of solutions of problems in No. 2;] E. S. Farrow, 111; Henry Gunder, 111, 112, 113, 114 and 118; Orra Garvin, 111; Flora Henderson, 111; William Hoover, 111, 112, 113, 114, 116 and 118; W. W. Johnson, 114 and 118; Christine Ladd, 111, 112, 114, and 118; Artemas Martin, 111, 113, 114 and 116; Dr. A. B. Nelson, 111, 113, 114 & 116; O. D. Oathout, 111, 114 and 116; Prof. J. Scheffer, 111, 112, 113, 114 and 118; Anna T. Snyder, 111 and 114; S. W. Salmon, 113, 116 and 118; E. B. Seitz, 116.

111. "Given $\frac{x+y}{x-y} + \frac{x-y}{x+y} = 3\frac{1}{3} \ldots . (1)$, $x^2 + y^2 = 45, \ldots \ldots (2)$ to be solved without the use of an auxiliary unknown."

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Clearing (1) of fractions we get

$$5(x^2 - y^2) = 3(x^2 + y^2), = 135 \text{ by } (2).$$

$$x^2 - y^2 = 27. \qquad (3)$$

From (2) and (3) we easily find

$$x = 6, \qquad y = 3.$$

112. "Wishing to know the height of a tower standing at the summit of a slope on the opposite side of the street, and not being able to leave my room, I measured the angles of elevation of the bottom and top of the tower 40°, 70°, respectively, and the angle of depression of the foot of the slope 40°; and a passer-by carried a tape-line across, giving me the distance from my point of observation to the foot of the slope, 50 feet. I knew the angle made by the face of the slope with the horizontal plane of the street to be 60°. From these data, I found the height of the tower, having given, $\log\tan 20°$ equal 9.651066, $\log\sin 50° = 9.884254$, $\log 10 = 1$, and $\log 171072 = 5.233188$."

SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.

Let A be the point of observation, [the reader can readily construct the figure from the description,] C the foot of the slope, B its top and T the top of the tower. The bisector BD, of the angle ABC, is perpendicular to AC, for $BDA = 180° - 80° - 10° = 90°$. $AD = 25$, $AB = 25 \div \sin 10°$, and because $ATB = 20°$ and $TAB = 30°$,

$$TB = \frac{25}{\sin 10°} \cdot \frac{\sin 30°}{\sin 20°} = 25(\cot 20° + \cot 10°), = 210.47 \text{ ft.}$$

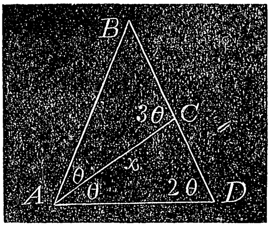
a solution shorter than that indicated by the problem.

[No solution of this question, employing only the data named in its announcement, has been received, though several of the solutions received use apparently more simple data.]

113. "The sides of a plane triangle are in arithmetical progression, common difference d, and the angle opposite the least side is one third of the angle opposite the greatest side; construct the triangle."

SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

Let the sides of the triangle ABC be $x - d$, x and $x + d$, and the angles opposite $x - d$ and $x + d$, θ and 3θ, respectively.



From A lay off CAD equal to θ and prolong AD till it meets BC produced in D. Then $BDA = BCA - CAD = 3\theta - \theta = 2\theta$ and $BAD = 2\theta$ by construction; therefore the triangle ABD is isosceles and hence $CD = (x + d) - (x - d) = 2d$. Because AC is the bisector of BAD, $AD = 2d\frac{x+d}{x-d}$

Now $AB\times AD = AC^2 + BC\times CD$ or $2d(x+d)^2 \div (x-d) = x^2 + 2d(x-d)$. Reducing, $2d[(x + d)^2 - (x - d)^2] = x^2(x - d)$ or $8d^2 = x(x - d)$, and this value of x is readily constructed.

---

114. "Required the shortest proof of the *Pons Asinorum*. (Eucl. 47, I.)"

SOLUTION BY DR. A. B. NELSON, DANVILLE, KENTUCKY.

By Quaternions, denoting the sides of the triangle in order by the vectors α, β, γ, α and β including the right angle, we have

$$\alpha + \beta = -\gamma, \text{ or } \alpha^2 + 2S\alpha\beta + \beta^2 = \gamma^2.$$

But, since α and β are perpendicular to each other, $S\alpha\beta = 0$. Hence, passing from vectors to lines, $a^2 + b^2 = c^2$. Q. E. D.

---

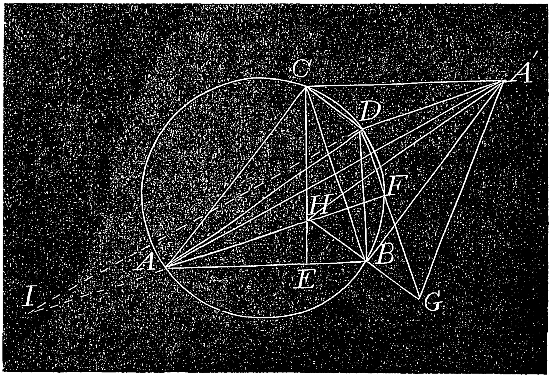
115. "Show, by a geometrical construction, that the results obtained by Miss Ladd and Prof. Chase, in their solutions of Prob. 106, are the same."

SOLUTION BY THE EDITOR.

Let the sides AB, AC and BC of the triangle ABC be represented by α, β, γ, respectively, and let the angle $BAC = \theta$.

Draw CA' equal and parallel to AB, join BA' and draw the diagonal AA'; then is $\alpha + \beta = AA'$ and $\alpha - \beta = \gamma = BC$.

Let a circle circumscribe the triangle ABC and draw its diameter AD. Draw AF and CE respectively perpendicular to BC & AB, and join DB, DC and DF.



Because ABD, ACD and AFD are angles in a semicircle they are all right angles, and therefore DB is parallel to CE and DF is parallel to CB.

Draw BH parallel and equal to CD, and produce HB to meet DF produced in G; then is $BG = BH = CD$. And because BH is parallel to CD it is perpendicular to AC, and therefore the point H is at the intersection of the two perpendiculars AF and CE. Hence the triangles ABH and $A'CD$, having two sides in one equal and parallel to two sides in the other, are identical, and therefore DA' is equal and parallel to AH.

Because BC and DG are parallel the alternate angles CBD and GDB are equal; and because $GDB = FDB = FAB$, and $CBD = CAD$, $\therefore$ $\angle HAB = \angle DAC = \angle HA'B = \angle GA'B$; $\therefore$ $\angle CA'D = \angle GA'B$. Therefore the angle $GA'D$ equals the angle CAE, and because AD is parallel to AH it is therefore perpendicular to DG and the triangles ACE and $A'GD$ are similar. Hence, as $DG = BC = \alpha - \beta$ and $\angle DA'G = \theta$, we have

$$A'D : (\alpha - \beta) :: \cos\theta : \sin\theta,$$

$$A'D = \frac{(\alpha - \beta)\cos\theta}{\sin\theta}.$$

Prof. Chase's answer is therefore represented by AD, the diagonal of the parallelogram $AA'DI$, while Miss Ladd's answer is represented also by AD, the diameter of a circle which circumscribes the quaternion $\beta \div \alpha$.

---

116. "A sector less than a semicircle is cut at random from a given circle, and a circle inscribed in it. Find the average area of this inscribed circle."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let O represent the center of the given circle, BOD the sector, and OA a radius of the given circle through M, the center of the inscribed circle; and let MN be a perpendicular on OD. Put $OA = r$, $MA = MN = x$, $\angle BOD = \theta$. Then $x = (r - x)\sin\frac{1}{2}\theta$, whence $x = r\sin\frac{1}{2}\theta \div (1 + \sin\frac{1}{2}\theta)$, and the required average

$$= \frac{1}{\pi}\int_0^\pi \pi x^2 d\theta = r^2\int_0^\pi \left(1 - \frac{1}{1 + \sin\frac{1}{2}\theta}\right)^2 d\theta$$

$$= r^2\int_0^\pi [1 - \tfrac{1}{2}\sec^2\tfrac{1}{4}(\pi - \theta)]^2 d\theta$$

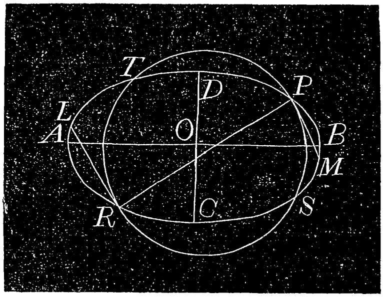
$$= r^2[\theta + 3\tan\tfrac{1}{4}(\pi - \theta) - \tfrac{1}{3}\tan^3\tfrac{1}{4}(\pi - \theta)]_0^\pi = r^2(\pi - \tfrac{8}{3}).$$

---

117. "Required the average area of all the acute-angled triangles that can be inscribed in a given ellipse."

SOLUTION BY R. J. ADCOCK, MONMOUTH, ILLINOIS.

Let $a^2y^2 + b^2x^2 = a^2b^2$, be the equation of the ellipse. From any point P in the first quadrant BD of the ellipse, draw any chord PR, and from its extremities the perpendiculars PM and RL. Upon PR as a diameter describe a circle intersecting the ellipse at S and T. Let the abscissas of the p'ts P, R, M, L, T, S, be respectsvely x', x'', x_1, x_2, x_3, x_4, and let the chord $PR = 2r$, p and p' being pependiculars upon the chord from any points in the elliptic arcs LT and SM, respectively, and dz, the elementary arc of the ellipse.



Then while PR is fixed we shall have

$$r\int_{x_2}^{x_3} pdz + r\int_{x_4}^{x_1} p'dz$$

= the sum of the areas of all the acute angled triangles that can be described on it.

Next regarding P as fixed and R movable, and integrating, after multiplying the preceding sum, now variable, by dz, we have

$$\int_{x''=-a}^{x'=a} r\left[\int_{x_2}^{x_3} pdz + \int_{x_4}^{x_1} p'dz\right]dz$$

= the sum of the areas of all the acute angled triangles having a common vertex P.

Multiplying again by dz, and integrating the preceding integral, now variable in consequence of P occupying all the different points in the quadrant BD, we get, finally,

$$4\int_{x'=0}^{x'=a}\left[\int_{x''=-a}^{x''=a} r\left(\int_{x_2}^{x_3} pdz + \int_{x_4}^{x_1} p'dz\right)dz\right]dz$$

= the sum of the areas of all the acute angled triangles that can be inscribed in the ellipse; which divided by

$$4\int_{x'=0}^{x'=a}\left[\int_{x''=-a}^{x''=a}\left(\int_{x_2}^{x_3} dz + \int_{x_4}^{x_1} dz\right)dz\right]dz,$$

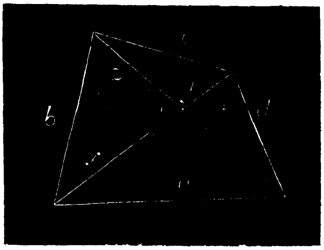
the whole number of acute triangles, gives the average required.

It should be observed that for each of the above integrals, except the last which is confined to the quadrant BD, the sum of two is to be used, one for the positive values of T and the other for the negative values.

---

118. "Find the general relation which exists between the four sides and the two diagonals of any quadrilateral. Consider particularly the case when the opposite sides are equal."

SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.



Let a, b, c, d represent the four sides of the quadrilateral, and f and g the two diagonals; also let x, y, z, t represent the four segments of the diagonals and α the angle which these diagonals make with each other. Then we can easily form the following equations;

$$a^2 = x^2 + t^2 + 2xt\cos\alpha, \; . \; . \; . \; (1)$$

$$b^2 = x^2 + z^2 - 2xz\cos\alpha, \; . \; . \; . \; (2)$$

$$c^2 = y^2 + z^2 + 2yz\cos\alpha \ldots (3), \qquad d^2 = y^2 + t^2 - 2yt\cos\alpha. \ldots (4)$$

Adding (1) and (3), and (2) and (4), and subtracting the latter sum from the former we obtain,

$$a^2 + c^2 - b^2 - d^2 = 2(xz + yz + xt + gt)\cos\alpha;$$

but $$xz + yz + xt + gt = (x + y)(z + t) = fg;$$

hence $$a^2 + c^2 - b^2 - d^2 = 2fg\cos\alpha,$$

and $$\text{sine}\,\alpha = \frac{1}{2fg}\sqrt{\left[4f^2g^2 - (a^2 + c^2 - b^2 - d^2)^2\right]}.$$

Denoting the area of the quadrilateral by A, we have

$$A = \tfrac{1}{2}fg\sin\alpha = \tfrac{1}{4}\sqrt{[4f^2g^2 - (a^2 + c^2 - b^2 - d^2)^2]},$$

whence we get the relation

$$4f^2g^2 = (a^2 + c^2 - b^2 - d^2)^2 + 16A^2.$$

This formula enables us to find a diagonal if the four sides and the other diagonal are given, for the area can easily be expressed by the four sides and one diagonal.

In case of a parallelogram we have $a = c$, $b = d$,

$$\therefore \quad 16A^2 = 4(2a^2b^2 + 2a^2g^2 + 2b^2g^2 - a^4 - b^4 - g^4),$$

consequently $4f^2g^2 = 4(a^2 - b^2)^2 + 4(2a^2b^2 + 2a^2g^2 + 2b^2g^2 - a^4 - b^4 - g^4)$,

or simplified, $\quad f^2 + g^2 = 2(a^2 + b^2)$: Hence—

In a parallelogram the sum of the squares of the two diagonals is equal to the sum of the squares of the four sides. This is a particular case of the more general theorem, first given by Euler, and called after him. "Euler's Theorem:"—*In any quadrilateral the sum of the squares of the diagonals is equal to the sum of the squares of the four sides increased by four times the square of the line connecting the middle points of the diagonals.*

If $\alpha = 90°$, we have $a^2 + c^2 = b^2 + d^2$. And by substitution we easily

obtain $$\left(\frac{f^2 + b^2 - c^2}{2f}\right)^2 + \left(\frac{g^2 - c^2 + d^2}{2g}\right)^2 = a^2.$$